Well-posedness of the Cauchy problem for the Maxwell-Dirac system in one space dimension (Mathematical Analysis in Fluid and Gas Dynamics)

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1 Introduction

In this note, we study the Cauchy problem of the Maxwell-Dirac (M-D) system in $1+1$ dimensions;

$$(-i\gamma^{\mu}\partial_{\mu}+m)\psi=A_{\mu}\gamma^{\mu}\psi, \tag{1.1}$$

$$\Box A_{\mu}=-\langle\gamma^{0}\gamma_{\mu}\psi,\psi\rangle, \tag{1.2}$$

$$\partial^{\mu}A_{\mu}=0, \tag{1.3}$$

$$\psi(0)=\psi_{0},\ A_{\mu}(0)=a_{\mu},\ \partial_{t}A_{\mu}(0)=\dot{a}_{\mu} \tag{1.4}$$

where $\partial_{0}=\partial_{t}$, $\partial_{1}=\partial_{x}$, $\Box=-\partial_{t}^{2}+\partial_{x}^{2}$, $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{C}^{2}$, $\psi=\psi(t,x)$ is a $\mathbb{C}^{2}$ valued unknown function, $A_{\mu}=A_{\mu}(t,x)$ are real valued unknown functions, and m is a nonnegative constant. We are concerned with the Minkowski space with the metric $g^{\mu\nu}=\mathrm{diag}(1,-1)$ and the summation convention is used for summing over repeated indices. Matrices γ^{μ} satisfy the conditions

$$\gamma^{\mu}\gamma^{\nu}+\gamma^{\nu}\gamma^{\mu}=2g^{\mu\nu}, \tag{1.5}$$

$$(\gamma^{0})^{*}=\gamma^{0},\ (\gamma^{1})^{*}=-\gamma^{1}. \tag{1.6}$$

The constraint (1.3) is the Lorenz gauge condition. The M-D system describes an electron self-interacting with its own electromagnetic field. The system in $1+1$ dimensions is the prototype model in the quantum field theory.

We put $\alpha^{0}=I_{2}$, $\alpha=\alpha^{1}=\gamma^{0}\gamma^{1}$, and $\beta=\gamma^{0}$, where I_{2} denotes the identity matrix of size 2. Matrices α^{μ}, β are Hermitian matrices and satisfy the conditions

$$(\alpha^{\mu})^{2}=\beta^{2}=I_{2},\quad \alpha^{1}\beta+\beta\alpha^{1}=0.$$

Then, (1.1) and (1.2) become

$$(-i\alpha^{\mu}\partial_{\mu}+m\beta)\psi=A_{\mu}\alpha^{\mu}\psi, \tag{1.7}$$

$$\Box A_{\mu}=-\langle\alpha_{\mu}\psi,\psi\rangle. \tag{1.8}$$

In the one dimensional case, the equations (1.2) and (1.3) require the initial data to satisfy the following two compatibility conditions:

$$\partial_x \dot{a}_1(x) = |\psi_0(x)|^2 + \partial_x^2 a_0(x), \quad \dot{a}_0(x) = \partial_x a_1(x). \tag{1.9}$$

The Lorenz gauge condition (1.3) restricts the behavior of the solutions at the spatial infinity, though wave equations have finite speed propagation. Indeed, if $\partial_x a_0$ and $\dot{a}_1$ vanish at $x = \pm\infty$, then (1.9) implies that

$$\int_{-\infty}^{\infty} |\psi_0|^2 = \|\psi_0\|_{L^2}^2 = 0,$$

which excludes the nontrivial case, this was pointed out in [24]. It is a difficulty of the one dimensional case. Let f be a real valued function in $C^\infty(\mathbb{R})$ satisfying the following assumption

$$f(x) = \frac{c_0}{2} x \text{ on } |x| \le \frac{2}{5}, \quad f(x) = \operatorname{sgn} x \cdot \frac{c_0}{2} \text{ on } |x| \ge \frac{3}{5},$$

$c_0 := \|\psi_0\|_{L^2}^2$. In this note, we consider the case $s \ge 0$ and the initial data $\dot{a}_1 - f$ vanishing at $\pm\infty$. This condition for the initial data $\dot{a}_1$ of the spatial infinity does not unnatural condition physically. Replacing $A_1(t,x)$ with $A_1(t,x) + tf(x)$, we rewrite (1.1)-(1.4) as follows.

$$(-i\alpha^\mu \partial_\mu + m\beta)\psi = A_\mu \alpha^\mu \psi + tf\alpha\psi, \tag{1.10}$$

$$\Box A_\mu = -\langle \alpha_\mu \psi, \psi \rangle - \mu t \partial_x^2 f, \tag{1.11}$$

$$\partial^\mu A_\mu = -t\partial_x f, \tag{1.12}$$

$$\psi(0) = \psi_0, \ A_\mu(0) = a_\mu, \ \partial_t A_\mu(0) = \dot{a}_0. \tag{1.13}$$

Remark 1.1. If (1.11) and (1.12) are satisfied by the initial datum, then the solution to M-D system also satisfies (1.12). Thus, we can remove (1.12) from the system.

The initial datum ψ_0, a_μ, and $\dot{a}_\mu$ of the Cauchy problem will be taken in a Sobolev space $H^s = H^s(\mathbb{R})$ defined by the norm

$$\|u\|_{H^s} := \|\langle \cdot \rangle^s \widehat{u}\|_{L^2},$$

where $\langle \cdot \rangle := (1 + |\cdot|^2)^{1/2}$ and $\widehat{u}$ denotes the Fourier transform of u. For $1+n$ dimensions, the M-D system with $m = 0$ is invariant under the scaling

$$\psi(t,x) \to \frac{1}{\lambda^{3/2}} \psi\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right), \quad A_\mu(t,x) \to \frac{1}{\lambda} A_\mu\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right),$$

hence the scaling invariant data space is

$$\psi_0 \in \dot{H}^{n/2-3/2}(\mathbb{R}^n), \ a_\mu \in \dot{H}^{n/2-1}(\mathbb{R}^n),$$

where $\dot{H}^s(\mathbb{R}^n)$ denotes a homogeneous Sobolev space. One does not expect the well-posedness below this regularity.

There are not many results on the 1 + 1 dimensional case unlike the higher dimensional case. Chadam [5] obtained the global existence of solution in $H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times L^2(\mathbb{R})$. In the case $m = 0$, Huh [12] proved the global well-posedness in $L^2(\mathbb{R}) \times C_b(\mathbb{R}) \times C_b(\mathbb{R})$. Note that the wave data a_μ and $\dot{a}_\mu$ are taken in the same space $C_b(\mathbb{R})$ and $\partial_t A_\mu \in C_b(\mathbb{R})$ is not proved in [12]. Usually, we assume that the regularity of $\dot{a}_\mu$ is one derivative less than a_μ, and for the well-posedness, we have to prove the solution stays in the same space as the initial data, which is called the "persistency". Recently, the well-posedness for the M-D system in $1+3$ and $1+2$ dimensions has intensively been studied by D'Ancona, Foschi and Selberg [7] and D'Ancona and Selberg [9] (see also [6]). Especially, the three dimensional result obtained by D'Ancona, Foschi, and Selberg [7] is optimal with respect to the scaling except for the critical case $L^2(\mathbb{R}^3) \times H^{1/2}((\mathbb{R}^3)$.

We describe two new ingredients of the proof by D'Ancona, Foschi, and Selberg [7] and the difference between the higher dimensional and the one dimensional cases. The first one is they have uncovered an additional null form in the Dirac equation. We here explain null forms and null form estimates. In the 3-dimension case, the quadratic forms in first derivatives

$$Q_0(f,g) = -\partial_t f \partial_t g + \sum_{j=1}^{3} \partial_j f \partial_j g,$$
$$Q_{\mu\nu}(f,g) = \partial_\mu f \partial_\nu g - \partial_\nu f \partial_\mu g, \ 0 \le \mu < \nu \le 3,$$

are said to be null forms. The space-time estimates for null forms were first proved in Klainerman and Machedon [13]. They were used to improve the classical local existence theorem for nonlinear wave equations with the null forms. Using the classical method, i.e., energy estimates and the embedding theorems, one can prove that the M-D system in $1+3$ dimensions is locally well-posed in $H^2(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$. Roughly speaking, the use of the Strichartz inequality allows us to improve classical local existence theorems by 1/2 derivative. However, the Strichartz inequality method does not take into account the special structure of the nonlinearities that come up in the equations. Using the null form estimates, Bournaveas [3] proved local well-posed in $H^{1/2+\varepsilon}(\mathbb{R}^3) \times H^{1+\varepsilon}(\mathbb{R}^3)$ for $\varepsilon > 0$. D'Ancona, Foschi, and Selberg [6, 7] have uncovered the full null structure which can not be seen directly. The null structure found in [6, 7] is not the usual bilinear null structure that may be seen in bilinear terms of each individual component equation of a system. But one can find the special property depends on the structure of the system as a whole. Hence, they call it system null structure. In the $1+1$ dimensional case, we can find the system null structure by employing the argument in [7]. Thus, our task is to prove the one dimensional null form estimates.

The second one is the fact that the M-D system in Lorenz gauge with space being

3-dimension or 2-dimension can be rewritten as the following system.

$$\nabla \cdot \boldsymbol{E} = \rho, \ \nabla \cdot \boldsymbol{B} = 0, \ \nabla \times \boldsymbol{E} + \partial_t \boldsymbol{B} = 0, \ \nabla \times \boldsymbol{B} - \partial_t \boldsymbol{E} = \boldsymbol{J},$$
$$(\alpha^\mu D_\mu + m\beta)\psi = 0,$$

where $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, $\boldsymbol{E} = \nabla A_0 - \partial_t \boldsymbol{A}$, $\boldsymbol{A} = (A_1, A_2, A_3)$, $D_\mu = \frac{1}{i}\partial_\mu - A_\mu$, $J^\mu = \langle \alpha^\mu \psi, \psi \rangle$, $\rho = J^0 = |\psi|^2$, and $\boldsymbol{J} = (J^1, J^2, J^3)$. From this expression, we may consider the M-D system in $1+n$ dimensions, $n \geq 2$ as the system of the fields $(\boldsymbol{B}, \boldsymbol{E})$ and the spinor ψ, instead of the potentials A_μ and the spinor ψ. In this case, the worst part of A_μ, that has no better structure, can be neglected. The observation plays a crucial role in the proof of [7] and [9]. On the other hand, in $1+1$ dimensions the electromagnetic fields $(\mathbf{B}, \mathbf{E})$ are not necessarily converted to the potential fields A_μ decaying near the spatial infinity. We directly consider the system of the potentials A_μ and the spinor ψ, and we must estimate the worst part of A_μ.

The M-D system has the charge conservation low;

$$\int |\psi(t)|^2 dx = \text{constant}.$$

It is natural and important to ask whether or not the global existence of the solution to the M-D system follows the charge conservation. Using this conservation, the global existence of solution was proved by [5], [10], and [12] for $1+1$ dimensions and by [9] for $1+2$ dimensions. In view of the scaling, $L^2(\mathbb{R}) \times H^{1/2}(\mathbb{R})$ is natural charge class. The problem with initial data in $L^2(\mathbb{R}) \times H^{1/2}(\mathbb{R})$ has been solved for the $1+2$ dimensional case, but it remains open in $1+1$ and $1+3$ dimensions.

We define the well-posedness in this note as follows.

Definition 1.1. *The Cauchy problem (1.10)-(1.13) is said to be locally well-posed in $H^s \times H^r$ if for any radius R there exists a time $T = T(R) > 0$ and a continuous flow map from $\{(\psi_0, a_\mu, \dot{a}_\mu) \in H^s \times H^r \times H^{r-1} : \|(\psi_0, a_\mu, \dot{a}_\mu)\|_{H^s \times H^r \times H^{r-1}} \leq R\}$ to $C([-T,T]; H^s) \times \big(C([-T,T]; H^r) \cap C^1([-T,T]; H^{r-1})\big)$.*

Remark 1.2. The following assertion is equivalent to Definition 1.1 : for every $\delta > 0$, there exists a $T > 0$ such that if $\|(\psi_0, a_\mu, \dot{a}_\mu)\|_{H^s \times H^r \times H^{r-1}} \leq \delta$ holds, the solution to M-D system on $[-T, T]$ exists, and for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\|(\psi_0, a_\mu, \dot{a}_\mu)\|_{H^s \times H^r \times H^{r-1}} \leq \delta$ holds, $\|(\psi, A_\mu, \partial_t A_\mu)\|_{C([-T,T]; H^s \times H^r \times H^{r-1})} \leq \varepsilon$, where $(\psi, A_\mu, \partial_t A_\mu)$ is the solution to M-D system with initial data $(\psi_0, a_\mu, \dot{a}_\mu)$.

We obtain the local well-posedness in $(0+, 1/2+)$, while the critical scaling regularity is $(-1, -1/2)$.

Theorem 1.2. *If $s > 0$, $s \leq r \leq \min(2s+1/2, s+1)$, $r > 1/2$, and $(s, r) \neq (1/2, 3/2)$, then (1.10)-(1.13) is locally well-posed in $H^s \times H^r$.*

In the proof of Theorem 1.2, we will pick out the worst part. The many restrictions in Theorem 1.2 comes from this part. Thus, we may suppose the well-posedness is broken by this part. We analyze this part in details and obtain the following theorems, which say Theorem 1.2 is optimal.

Theorem 1.3. *Suppose* $0 \leq s < 1/2$, $r > \max(2s+1/2, 1/2)$. *Then there exist sequences* $\{u_N\} \subset \mathscr{S}(\mathbb{R})$ *and* $t_N \searrow 0$ *such that* $\|u_N\|_{H^s} \to 0$, *as* $N \to \infty$, *and the corresponding solution* $(\psi_N, A_{\mu,N})$ *to (1.10)-(1.11) with initial data* $\left(\binom{u_N}{0}, 0, 0\right)$ *satisfies*

$$\|A_{0,N}(t_N)\|_{H^r} \to \infty, \quad as\ N \to \infty.$$

Remark 1.3. The ill-posedness appearing in Theorem 1.3 is referred to as norm inflation. It says that the flow map of (1.10)-(1.13) fails to be continuous at 0, and fails to be bounded in a neighborhood of 0. In the case of nonlinear operator, the notions of boundedness and continuity are not equivalent.

Theorem 1.4. *Suppose* $r < s$ *or* $r > s+1$ *or* $r \leq 1/2$ *or* $s = 1/2$, $r \geq 3/2$. *Then for any* $T > 0$, *the flow map of (1.10)-(1.13), as a map from the unit ball centered at* 0 *in* $H^s \times H^r \times H^{r-1}$ *to* $C([-T,T]; H^s) \times \left(C([-T,T]; H^r) \cap C^1([-T,T]; H^{r-1})\right)$, *fails to be* C^2.

We note that the M-D system does not have better structure than the Dirac-Klein-Gordon (D-K-G) system.

$$\begin{aligned} (-i\gamma^\mu \partial_\mu + M)\psi &= \varphi\psi, \\ (-\Box + m^2)\varphi &= \langle \gamma^0 \psi, \psi \rangle, \end{aligned}$$

where $\psi = \psi(t,x)$ is a $\mathbb{C}^2$ valued unknown function, $\varphi = \varphi(t,x)$ is a real valued unknown function, m and M are nonnegative constants. Machihara, Nakanishi, and Tsugawa [16] proved the local well-posedness for D-K-G in $H^s(\mathbb{R}) \times H^r(\mathbb{R})$, provided that s and r satisfy the conditions $s > -1/2$ and $|s| \leq r \leq s+1$. The difference between Theorem 1.2 and the result in [16] comes from the structure of the right hand side of each second equation. The right hand side of (1.2) with $\mu = 0$ is the square of ψ, which is the worst part. This part has no null structure and proving the local well-posedness for small (s,r) is a difficult problem. The part that breaks down the proof of the well-posedness may imply the ill-posedness. In our case, the norm inflation comes from this part.

Remark 1.4. Theorem 1.4 does not imply the ill-posedness but precludes proofs of the well-posedness by the contraction argument. Indeed, if the contraction argument works, the flow map proves to be C^∞ in most cases.

This note is organized as follows. In Section 2 we prove the well-posedness results. In Section 3 we prove the ill-posedness results.

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2 Local well-posedness

As in [7], we decompose A_μ as follows:

$$A_\mu = W(t)[a_\mu, \dot{a}_\mu] + A_\mu^{\text{inh.}} - \mu(tf - W(t)[0,f]),$$
$$A_\mu^{\text{inh.}} = -\Box^{-1}\langle \alpha_\mu \psi, \psi\rangle.$$

Here we use the notations $W(t)[a,b]$ and $\Box^{-1}F$ for the solution of the homogeneous wave equation with initial data a, b and the solution of the inhomogeneous wave equations $\Box u = F$ with vanishing data at time $t = 0$, respectively. We decompose the spinor as

$$\psi = \psi_+ + \psi_-, \ \psi_\pm \equiv \Pi_\pm \psi, \tag{2.1}$$

where $\Pi_\pm \equiv \Pi(\pm\partial_x/i)$ is the multiplier whose symbol is the Dirac Projection

$$\Pi(\xi) := \frac{1}{2}(I_2 + \widehat{\xi}\alpha), \ \widehat{\xi} := \frac{\xi}{|\xi|}.$$

Note the identities

$$I_2 = \Pi(\xi) + \Pi(-\xi), \ \widehat{\xi}\alpha = \Pi(\xi) - \Pi(-\xi), \ \beta\Pi(\xi) = \Pi(-\xi)\beta, \tag{2.2}$$
$$\Pi(\xi)^* = \Pi(\xi), \ \Pi(\xi)^2 = \Pi(\xi), \ \Pi(\xi)\Pi(-\xi) = 0. \tag{2.3}$$

We see that (1.10) splits into two equations:

$$(-i\partial_t \pm |\partial_x|)\psi_\pm = -m\beta\psi_\mp + \Pi_\pm((A_\mu^{\text{hom.}} + \mu W(t)[0,f])\alpha^\mu\psi - \mathcal{N}(\psi,\psi,\psi)). \tag{2.4}$$

According to the linear part of (2.4), we define the following function spaces.

Definition 2.1. *For $s, b \in \mathbb{R}$, $X_\pm^{s,b}$ is the completion of the Schwartz space $\mathscr{S}(\mathbb{R}^{1+1})$ with respect to the norm*

$$\|u\|_{X_\pm^{s,b}} = \|\langle\xi\rangle^s\langle\tau \pm |\xi|\rangle^b \widetilde{u}(\tau,\xi)\|_{L^2_{\tau,\xi}},$$

where $\widetilde{u}(\tau,\xi)$ denotes the time-space Fourier transform of $u(t,x)$.

Definition 2.2. *For $s, b \in \mathbb{R}$, $H^{s,b}$ and $\mathcal{H}^{s,b}$ are the completion of $\mathscr{S}(\mathbb{R}^{1+1})$ with respect to the norm*

$$\|u\|_{H^{s,b}} := \|\langle\xi\rangle^s\langle|\tau| - |\xi|\rangle^b \widetilde{u}\|_{L^2_{\tau,\xi}},$$
$$\|u\|_{\mathcal{H}^{s,b}} := \|u\|_{H^{s,b}} + \|\partial_t u\|_{H^{s-1,b}},$$

respectively.

These spaces are introduced by Bourgain [2] and Klainerman and Machedon [14].

Remark 2.1. The norm $\|u\|_{\mathcal{H}^{s,b}}$ equivalent to $\|\langle\xi\rangle^{s-1}\langle|\tau|+|\xi|\rangle\langle|\tau|-|\xi|\rangle^{b}\widetilde{u}\|_{L^2_{\tau,\xi}}$.

Remark 2.2. For $b > 1/2$, we have $X^{s,b}_{\pm} \hookrightarrow C(\mathbb{R}; H^s)$ and $\mathcal{H}^{s,b} \hookrightarrow C(\mathbb{R}; H^s) \cap C^1(\mathbb{R}; H^{s-1})$. The dual spaces of $X^{s,b}_{\pm}$ and $H^{s,b}$ are $X^{-s,-b}_{\pm}$ and $H^{-s,-b}$, respectively.

By a standard argument (see, for instance, [6] or [7]) the problem obtaining closed estimates for the iterates reduces to proving the nonlinear estimates

$$\|\Pi_{\pm_2}(A^{\text{hom.}}_{\mu}\alpha^{\mu}\psi_{\pm_1})\|_{X^{s,-1/2+2\varepsilon}_{\pm_2}(S_T)} \lesssim \mathcal{I}_0\|\psi\|_{X^{s,1/2+\varepsilon}_{\pm_1}(S_T)}, \tag{2.5}$$

$$\|\Pi_{\pm_2}(W(t)[0,f]\alpha\psi_{\pm_1})\|_{X^{s,-1/2+2\varepsilon}_{\pm_2}(S_T)} \lesssim \mathcal{I}_0\|\psi\|_{X^{s,1/2+\varepsilon}_{\pm_1}(S_T)}, \tag{2.6}$$

$$\|\Pi_{\pm_4}\Box^{-1}\langle\alpha_{\mu}\psi_{\pm_1},\psi_{\pm_2}\rangle\alpha^{\mu}\psi_{\pm_3}\|_{X^{s,-1/2+2\varepsilon}_{\pm_4}(S_T)} \lesssim \prod_{j=1}^{3}\|\psi\|_{X^{s,1/2+\varepsilon}_{\pm_j}(S_T)}, \tag{2.7}$$

$$\|\langle\alpha_{\mu}\psi,\psi\rangle\|_{H^{r-1,-1/2+2\varepsilon}(S_T)} \lesssim \|\psi\|^2_{X^{s,1/2+\varepsilon}(S_T)}, \tag{2.8}$$

where $\mathcal{I}_0 = \|(\psi_0, a_{\mu}, \dot{a}_{\mu})\|_{H^s\times H^r\times H^{r-1}}$. We omit the details for the proof these estimates. Since the null structure plays crucial role in the proof, we only consider the null form estimates.

We define

$$\theta_{jk} = \theta(e_j, e_k) = \begin{cases}1, & e_je_k < 0,\\ 0, & e_je_k > 0\end{cases} = \begin{cases}1, & \pm_j = \pm_k, \xi_j\xi_k < 0 \text{ or } \pm_j \neq \pm_k, \xi_j\xi_k > 0,\\ 0, & \pm_j = \pm_k, \xi_j\xi_k > 0 \text{ or } \pm_j \neq \pm_k, \xi_j\xi_k < 0.\end{cases}$$

Remark 2.3. In higher dimensions, θ_{ij} denotes the angle between e_i and e_j, and works system null structure (see [7, 9]).

We use the notation

$$\mathcal{F}[\mathfrak{B}_{\sigma}(u_1,u_2)](X_0) = \iint \sigma(X_1,X_2)\widetilde{u_1}(X_1)\overline{\widetilde{u_2}(X_2)}d\mu^{12}_{X_0}.$$

The following Proposition is the 1-dimensional null form estimates.

Propositon 2.3. *Suppose $s_0, s_1, s_2 \in \mathbb{R}$, $b_0, b_1, b_2 \geq 0$. We define $A := b_0 + b_1 + b_2$, $B = \min(b_0, b_1, b_2)$, and $s = s_0 + s_1 + s_2)$. If*

$$\begin{gathered} s_0 + s_1 \geq 0,\ s_0 + s_2 \geq 0,\ A > 1/2,\\ s_1 + s_2 + A > 1/2,\ s + A > 1,\\ s_1 + s_2 + B \geq 0,\ s + B \geq 1/2,\end{gathered}$$

we then have

$$\|\mathfrak{B}_{\theta_{12}}(u_1,u_2)\|_{X^{-s_0,-b_0}_{\pm_0}} \lesssim \|u_1\|_{X^{s_1,b_1}_{\pm_1}}\|u_2\|_{X^{s_2,b_2}_{\pm_2}}. \tag{2.9}$$

If

$$\begin{gathered} s_0 + s_2 \geq 0,\ s_1 + s_2 \geq 0,\ A > 1/2,\\ s_0 + s_1 + A > 1/2,\ s + A > 1,\\ s_0 + s_1 + B \geq 0,\ s + B \geq 1/2,\end{gathered}$$

we then have

$$\|\mathfrak{B}_{\theta_{01}}(u_1,u_2)\|_{X^{-s_0,-b_0}_{\pm_0}} \lesssim \|u_1\|_{X^{s_1,b_1}_{\pm_1}} \|u_2\|_{X^{s_2,b_2}_{\pm_2}}. \tag{2.10}$$

Proof. We only prove (2.9), because the proof of (2.10) is similar. By a duality argument, (2.9) is equivalent to

$$\left| \iint \frac{\theta_{12}\langle F_1(X_1), F_2(X_2)\rangle F_0(X_0)}{\langle\sigma_0\rangle^{b_0}\langle\sigma_1\rangle^{b_1}\langle\sigma_2\rangle^{b_2}\langle\xi_0\rangle^{s_0}\langle\xi_1\rangle^{s_1}\langle\xi_2\rangle^{s_2}} d\mu^{12}_{X_0} dX_0 \right| \lesssim \|F_1\|\|F_2\|\|F_0\|, \tag{2.11}$$

where

$$\widetilde{u}_j(X_j) = \frac{F_j(X_j)}{\langle\sigma_j\rangle^{b_j}\langle\xi_j\rangle^{s_j}}, \ \sigma_j = \tau_j \pm_j |\xi_j|, \ X_j = (\tau_j, \xi_j), \ F_j \in L^2.$$

If $\theta_{12} = 0$, (2.9) is trivial. Assuming $\theta_{12} \neq 0$, we have $\pm_1 \neq \pm_2$, $\xi_1\xi_2 > 0$ or $\pm_1 = \pm_2$, $\xi_1\xi_2 < 0$. We only consider the case $\pm_1 \neq \pm_2$, $\xi_1\xi_2 > 0$, since the other case can be handled similarly.

We then have $|\xi_0| = ||\xi_1| - |\xi_2||$ and

$$\sigma_0 - \sigma_1 + \sigma_2 = \pm_0|\xi_0| \mp_1 (|\xi_1| + |\xi_2|) = \pm_0||\xi_1| - |\xi_2|| \mp_1 (|\xi_1| + |\xi_2|).$$

Thus we get $\min(|\xi_1|, |\xi_2|) \lesssim \max(|\sigma_0|, |\sigma_1|, |\sigma_2|)$. Since $X_0 = X_1 - X_2$, one of the following must hold:

$$|\xi_0| \ll |\xi_1| \sim |\xi_2|, \tag{2.12}$$

$$|\xi_0| \sim \max(|\xi_1|, |\xi_2|) \geq \min(|\xi_1|, |\xi_2|). \tag{2.13}$$

In the case (2.12), (2.11) reduces to

$$\iint \frac{|F_0(X_0)F_1(X_1)F_2(X_2)|}{\langle\sigma_0\rangle^{b_0}\langle\sigma_1\rangle^{b_1}\langle\sigma_2\rangle^{b_2}\langle\xi_0\rangle^{s_0}\langle\xi_1\rangle^{s_1+s_2}} d\mu^{12}_{X_0} dX_0 \lesssim \|F_0\|\|F_1\|\|F_2\|. \tag{2.14}$$

We consider the case $\langle\sigma_0\rangle \geq \langle\sigma_1\rangle \geq \langle\sigma_2\rangle$. We get $\langle\xi_1\rangle \lesssim \langle\sigma_0\rangle$. If $b_1 + b_2 > 1/2$, we then have

$$\begin{aligned}
\text{L.H.S of (2.14)} &\lesssim \iint \frac{|F_0(X_0)F_1(X_1)F_2(X_2)|}{\langle\sigma_2\rangle^{b_1+b_2-}\langle\xi_0\rangle^{s_0}\langle\xi_1\rangle^{s_1+s_2+b_0+}} d\mu^{12}_{X_0} dX_0 \\
&\lesssim \left\| \mathcal{F}^{-1}\left[\frac{F_0}{\langle\xi_0\rangle^{s+b_0+}}\right]\right\|_{L^2_t L^\infty_x} \|F_1\|_{L^2_{t,x}} \left\|\mathcal{F}^{-1}\left[\frac{F_2}{\langle\sigma_2\rangle^{b_1+b_2-}}\right]\right\|_{L^\infty_t L^2_x} \\
&\lesssim \|F_0\|\|F_1\|\|F_2\|
\end{aligned}$$

where we have used Hölder's inequality, Young's inequality, and Sobolev's inequality. If $b_1 + b_2 \leq 1/2$, dividing $b_0 = (1/2 - b_1 - b_2) + (A - 1/2)$, we then have

$$\begin{aligned}
\text{L.H.S. of (2.14)} &\lesssim \iint \frac{|F_0(X_0)F_1(X_1)F_2(X_2)|}{\langle\sigma_2\rangle^{1/2+}\langle\xi_0\rangle^{s_0}\langle\xi_1\rangle^{s_1+s_2+A-1/2-}} d\mu^{12}_{X_0} dX_0 \\
&\lesssim \left\| \mathcal{F}^{-1}\left[\frac{F_0}{\langle\xi_0\rangle^{s+A-1/2-}}\right]\right\|_{L^2_t L^\infty_x} \|F_1\|_{L^2_{t,x}} \left\|\mathcal{F}^{-1}\left[\frac{F_2}{\langle\sigma_2\rangle^{1/2+}}\right]\right\|_{L^\infty_t L^2_x} \\
&\lesssim \|F_0\|\|F_1\|\|F_2\|.
\end{aligned}$$

The remaining cases are handled similarly. □

If the bilinear form has no null structure, the following estimate holds. We omit the proof of Proposition 2.4, since it is similar to Proposition 2.3.

Propositon 2.4. *Suppose* $s_0, s_1, s_2 \in \mathbb{R}$, $b_0, b_1, b_2 \geq 0$, *and* $b_0 + b_1 + b_2 > 1/2$. *If*

$$s_0 + s_1 + s_2 \geq \max(s_0, s_1, s_2), \;\; s_0 + s_1 + s_2 \geq 1/2,$$

or

$$s_0 + s_1 + s_2 > \max(s_0, s_1, s_2), \;\; s_0 + s_1 + s_2 \geq 1/2,$$

and we do not allow both to be equalities, we then have

$$\|u_1 \overline{u_2}\|_{X^{-s_0,-b_0}_{\pm_0}} \lesssim \|u_1\|_{X^{s_1,b_1}_{\pm_1}} \|u_2\|_{X^{s_2,b_2}_{\pm_2}} \tag{2.15}$$

for all $u_j \in X^{s_j,b_j}_{\pm_j}$, $j = 1, 2$.

Remark 2.4. By the null structure, Proposition 2.3 permits $s_0 + s_1 + s_2 < 1/2$, while Proposition 2.4 requires $s_0 + s_1 + s_2 > 1/2$. Roughly speaking, in Proposition 2.3, we can replace s_j by $s_j + b_j$.

3 Ill-posedness

Since all representations of operators satisfying (1.5) and (1.6) are unitary equivalent, we may choose

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.1}$$

for calculation.

The following statement follows from Theorem 1.2. Let $0 < \varepsilon \ll 1$ and let s and r satisfy $0 < s < 1$ and $\max(s, 1/2) < r < \min(2s + 1/2, s + 1)$. For $(\psi_0, a_\mu, \dot{a}_\mu) \in H^s \times H^r \times H^{r-1}$, there exist $T = T(\|(\psi_0, a_\mu, \dot{a}_\mu)\|_{H^s \times H^r \times H^{r-1}}) \in (0, 1]$ and the solution of M-D system with initial data $(\psi_0, a_\mu, \dot{a}_\mu)$ satisfies

$$\|\psi\|_{X^{s,1/2+\varepsilon}(S_T)} \leq C\|\psi_0\|_{H^s}, \tag{3.2}$$

$$\|A_\mu\|_{\mathcal{H}^{r,1/2+\varepsilon}(S_T)} \leq C(\|(a_\mu, \dot{a}_\mu)\|_{H^r \times H^{r-1}} + \|\psi_0\|^2_{H^s}). \tag{3.3}$$

3.1 Outline of the proof of Theorem 1.3

Let S_m be the free evolution operator of the Dirac equation expressed as $S_m(t) := \cos(\langle \partial_x \rangle_m t) + (\gamma^1 \partial_x - im)\gamma^0 \frac{\sin(\langle \partial_x \rangle_m t)}{\langle \partial_x \rangle_m}$, where $\langle \partial_x \rangle_m := (m^2 - \partial_x^2)^{1/2}$. By (3.1), we get

$$S_m(t) = \begin{pmatrix} \cos(\langle\partial_x\rangle_m t) + \frac{\partial_x}{\langle\partial_x\rangle_m}\sin(\langle\partial_x\rangle_m t) & -\frac{im}{\langle\partial_x\rangle_m}\sin(\langle\partial_x\rangle_m t) \\ -\frac{im}{\langle\partial_x\rangle_m}\sin(\langle\partial_x\rangle_m t) & \cos(\langle\partial_x\rangle_m t) - \frac{\partial_x}{\langle\partial_x\rangle_m}\sin(\langle\partial_x\rangle_m t) \end{pmatrix}. \tag{3.4}$$

We put

$$W(t) := \frac{\sin(t\sqrt{-\partial_x^2})}{\sqrt{-\partial_x^2}},$$

which is the free evolution operator of the wave equation. We set

$$\widehat{u}_N(\xi) = N^{-2s+r/2-3/4}(\chi_{[N,N+N^{2s-r+3/2}]}(\xi) + \chi_{[-N-N^{2s-r+3/2},-N]}(\xi)),$$

where χ_A is the characteristic function of A. Then we have

$$\|u_N\|_{H^{s'}} \leq N^{-2s+r/2-3/4}N^{s'+s-r/2+3/4} = N^{s'-s}. \tag{3.5}$$

We split the proof into four steps.

Step 1. *We now prove*

$$\left\| \int_0^t W(t-s)|S_m(s)\psi_{0,N}|^2 ds \right\|_{H^r} \gtrsim tN^{\sigma}, \ \sigma := -s + r/2 - 1/4 > 0$$

for $t \gtrsim 1/N$. Thus the desired result holds provided $u_{0,N}$ is replaced by $S_m(t)\psi_{0,N}$, where $\psi_{0,N} = \binom{u_N}{0}$.

Without loss of generality, we may consider the case $m = 1$. By a direct calculation, we have

$$\widehat{|S_1(t)\psi_{0,N}|^2}(\xi) = \int_{\xi_1+\xi_2=\xi} A(\xi_1,\xi_2)\widehat{u}_N(\xi_1)\widehat{u}_N(\xi_2) + \int_{\xi_1+\xi_2=\xi} B(\xi_1,\xi_2)\widehat{u}_N(\xi_1)\widehat{u}_N(\xi_2),$$

$$A(\xi_1,\xi_2) = \left(\cos(\langle\xi_1\rangle t) + i\frac{\xi_1}{\langle\xi_1\rangle}\sin(\langle\xi_1\rangle t)\right)\left(\cos(\langle\xi_2\rangle t) + i\frac{\xi_2}{\langle\xi_2\rangle}\sin(\langle\xi_2\rangle t)\right),$$

$$B(\xi_1,\xi_2) = \frac{\sin(\langle\xi_1\rangle t)}{\langle\xi_1\rangle}\frac{\sin(\langle\xi_2\rangle t)}{\langle\xi_2\rangle}.$$

We divide A into the reading term $e^{i(|\xi_1|+|\xi_2|)t}$ and the remainder $M(\xi_1,\xi_2) := A(\xi_1,\xi_2) - e^{i(|\xi_1|+|\xi_2|)t}$. Restricting ξ to the region $2N \leq |\xi| \leq 2N + 2N^{1-2\sigma}$, by symmetry, we only consider the case $\xi_1,\xi_2 \in [N, N+N^{1-2\sigma}]$. Then we get

$$\widehat{|u_N|^2}(\xi) = N^{4\sigma-r-1/2}h(\xi),$$

where

$$h(\xi) = \begin{cases} \xi - 2N, & \xi \in [2N, 2N+N^{1-2\sigma}], \\ -\xi + 2N + 2N^{1-2\sigma}, & \xi \in [2N+N^{1-2\sigma}, 2N+2N^{1-2\sigma}], \\ 0, & \text{otherwise.} \end{cases}$$

Thus we have

$$\int_0^t \frac{\sin((t-s)\xi)}{\xi} e^{i\xi s} h(\xi) ds = -\frac{1}{4\xi^2} h(\xi) e^{it\xi}(e^{-2it\xi} - 1 + 2it\xi).$$

For $|t\xi| \gtrsim 1$, we get

$$\left| \int_0^t \frac{\sin((t-s)\xi)}{\xi} e^{i\xi s} h(\xi) ds \right| \gtrsim \frac{h(\xi)}{\xi} t.$$

We obtain

$$\left(\int_{2N}^{2N+2N^{2s-r+3/2}} \langle \xi \rangle^{2r} \left| \int_0^t \frac{\sin((t-s)\xi)}{\xi} e^{i\xi s} \widehat{|u_{0,N}|^2}(\xi) ds \right|^2 d\xi \right)^{1/2} \gtrsim tN^{\sigma}.$$

Since $M(\xi_1, \xi_2) \lesssim t/N$, for $2N \le |\xi| \le 2N + 2N^{1-2\sigma}$, we have

$$\left| \int_0^t \frac{\sin((t-s)\xi)}{\xi} \int_{\xi_1+\xi_2=\xi} M(\xi_1,\xi_2) \widehat{u}_{0,N}(\xi_1) \widehat{u}_{0,N}(\xi_2) dt' \right| \lesssim t^2 N^{4\sigma-r-5/2} h(\xi),$$

which completes the proof of Step 1.

Step 2. *When* $0 < s < 1/2$ *and* $2s + 1/2 < r < \min(14s/11 + 19/22, 14s/3 + 1/2)$, *we prove*

$$\left\| \int_0^t W(t-s)(|\psi_N(s)|^2 - |S_m(s)\psi_{0,N}|^2) ds \right\|_{L_t^\infty H^r(S_T)} \lesssim N^{\sigma/2}.$$

Since

$$|\psi_N(t)|^2 - |S_m(t)\psi_{0,N}|^2 = |\psi_N(t) - S_m(t)\psi_{0,N}|^2 + 2\Re\langle \psi_N(t) - S_m(t)\psi_{0,N}, S_m(t)\psi_{0,N}\rangle$$

and $H^{r,1/2+\varepsilon}(S_T) \hookrightarrow L_t^\infty H^r(S_T)$, it suffices to show that

$$\left\| \int_0^t W(t-s)|\psi_N(s) - S_m(s)\psi_{0,N}|^2 ds \right\|_{H^{r,1/2+\varepsilon}(S_T)} \lesssim N^{\sigma/2}, \tag{3.6}$$

$$\left\| \int_0^t W(t-s)\langle \psi_N(s) - S_m(s)\psi_{0,N}, S_m(s)\psi_{0,N}\rangle ds \right\|_{H^{r,1/2+\varepsilon}(S_T)} \lesssim N^{\sigma/2}. \tag{3.7}$$

We only prove (3.6), because (3.7) can be handled similarly. We put

$$s_1 = 7s/4 - 3r/8 + 3/16, \; s_2 = -7s/4 + 11r/8 - 11/16.$$

From the conditions in Step 2, $0 < s_1 < s < s_2 < 1/2$. By Proposition 2.3,

$$\|\Box^{-1}\langle \alpha^\mu\psi, \psi\rangle \alpha_\mu \psi\|_{X_\pm^{s_2,-1/2+\varepsilon}} \lesssim \|\psi\|^2_{X^{s_1,1/2+\varepsilon}} \|\psi\|_{X^{s_2,1/2+\varepsilon}}. \tag{3.8}$$

Thus, we have

$$\begin{aligned}
&\|\psi_N - S_m(t)\psi_{0,N}\|_{X^{s_2,1/2+\varepsilon}(S_T)} \le C\|(A_\mu\alpha^\mu + tf\alpha)\psi\|_{X^{s_2,-1/2+\varepsilon}(S_T)} \\
&\le C\|\Box^{-1}\langle \alpha^\mu\psi_N, \psi_N\rangle \alpha_\mu\psi_N\|_{X_\pm^{s_2,-1/2+\varepsilon}(S_T)} + \|W(t)[0,f]\psi_N\|_{X^{s_2,-1/2+\varepsilon}(S_T)} \\
&\le C(\|\psi_N\|^2_{X^{s_1,1/2+\varepsilon}(S_T)} + \|f\|_{C^{1/2}})\|\psi_N\|_{X^{s_2,1/2+\varepsilon}(S_T)} \\
&\le C(\|\psi_N\|^2_{X^{s_1,1/2+\varepsilon}(S_T)} + \|f\|_{C^{1/2}}) \\
&\quad \times (\|\psi_N - \varphi S_m(t)\psi_{0,N}\|_{X^{s_2,1/2+\varepsilon}(S_T)} + \|\varphi S_m(t)\psi_{0,N}\|_{X^{s_2,1/2+\varepsilon}(S_T)}).
\end{aligned}$$

Since $\|\psi_N\|_{X^{s_1,1/2+\varepsilon}(S_T)} \lesssim \|\psi_{0,N}\|_{H^{s_1}} \lesssim N^{s_1-s}$ and $\|f\|_{C^{1/2}} \lesssim N^{-s}$, provided N is taken large enough, we get

$$\|\psi_N - \varphi S_m(t)\psi_{0,N}\|_{X^{s_2,1/2+\varepsilon}(S_T)} \le CN^{2(s_1-s)}\|\varphi S_m(t)\psi_{0,N}\|_{X^{s_2,1/2+\varepsilon}(S_T)}.$$

By the linear estimates (see [20]), Proposition 2.4 with $s_0 = 1 - r$ and $b_0 = 1/2 - 2\varepsilon$, (3.2) with $(s_j, s_j + 1/2)$, $j = 1, 2$, we obtain

$$\begin{aligned}
\text{L.H.S. of (3.6)} &\lesssim \|\langle \psi_N - S_m(t)\psi_{0,N}, \psi_N - S_m(t)\psi_{0,N}\rangle\|_{H^{r-1,-1/2+2\varepsilon}(S_T)} \\
&\lesssim \|\psi_N - S_m(t)\psi_{0,N}\|_{X_\pm^{s_1,1/2+\varepsilon}(S_T)} \|\psi_N - \varphi S_m(t)\psi_{0,N}\|_{X_\pm^{s_2,1/2+\varepsilon}(S_T)} \\
&\lesssim \|\psi_{0,N}\|_{H^{s_1}} N^{2(s_1-s)} \|\psi_{0,N}\|_{H^{s_2}} \lesssim N^{-4s+3s_1+s_2} = N^{\sigma/2}.
\end{aligned}$$

Step 3. *We obtain*

$$\|A_{0,N}(t)\|_{H^r} \gtrsim tN^\sigma$$

if $0 < s < 1/2$ *and* $2s + 1/2 < r < \min(14s/11 + 19/22, 14s/3 + 1/2)$, *and* $t \gtrsim 1/N$.

For, by Steps 1 and 2, we have that

$$\begin{aligned}
\|A_{0,N}(t)\|_{H^r} \ge &\left\| \int_0^t W(t-s)|S_m(s)\psi_{0,N}|^2 ds \right\|_{H^r} \\
&- \left\| \int_0^t W(t-s)(|\psi_N(s)|^2 - |\varphi(s)S_m(s)\psi_{0,N}|^2) ds \right\|_{L_t^\infty H^r(S_T)} \gtrsim tN^\sigma.
\end{aligned}$$

Step 4. *When* $0 \le s < 1/2$ *and* $r > 2s + 1/2$, *we have*

$$\|A_{0,N}(t)\|_{H^r} \ge CtN^\alpha$$

for some $\alpha > 0$ *and* $t \gtrsim 1/N$.

Indeed, let r' be such that $r' \le r$ and r' satisfy the conditions in Step 3, and let s' be such that $0 < s' < r'/2 - 1/4$, i.e., $r' > 2s' + 1/2$. From $\|\psi\|_{H^s} \le \|\psi\|_{H^{s'}}$, appealing the conclusion of Step 3 with s and r replaced by s' and r', we obtain

$$\|A_{0,N}\|_{H^r} \ge \|A_{0,N}\|_{H^{r'}} \gtrsim tN^{-s'+r'/2-1/2}.$$

3.2 Outline of the proof of Theorem 1.4

In the proof of Theorem 1.4, we can neglect the mass term (see, for instance, [16]). In this subsection, we abbreviate S_0 to S. We prove that if (s,r) satisfy the assumptions of Theorem 1.4, there exist a sequence $(\psi_{0,N}, a_{\mu,N}, \dot{a}_{\mu,N})$ satisfying

$$\|(\psi_{0,N}, a_{\mu,N}, \dot{a}_{\mu,N})\|_{H^s \times H^r \times H^{r-1}} \lesssim 1,$$

but $\|\psi_N^{(2)}\|_{H^s}$ or $\|A_{0,N}^{(2)}\|_{H^r}$ is unbounded, where $\psi_N^{(1)}(t) = S(t)\psi_{0,N}$, $A_{\mu,N}^{(1)}(t) = W(t)(a_{\mu,N}, \dot{a}_{\mu,N})$,

$$\psi_N^{(2)}(t) = -i\int_0^t S(t-s)(A_{\mu,N}^{(1)}(s)\alpha^\mu \psi_N^{(1)}(s))ds,$$
$$A_{\mu,N}^{(2)}(t) = -i\int_0^t W(t-s)\langle \alpha_\mu \psi_N^{(1)}(s), \psi_N^{(1)}(s)\rangle ds.$$

We only consider the case $s \in \mathbb{R}$ and $r = 1/2$. Define $a_{1,N} = \dot{a}_{0,N} = \dot{a}_{1,N} = 0$,

$$\widehat{u_N}(\xi) = N^{-2s-1/2}\chi_{[N^2-N,N^2+N]}(\xi),$$
$$\widehat{a_{0,N}}(\xi) = (\log N)^{-1/2}\langle\xi\rangle^{-1}\chi_{[1,N]}(\xi).$$

Since

$$\begin{aligned}\int_{N^2}^{N^2+N} |\widehat{a_{0,N}} * \widehat{u_N}(\xi)| d\xi &\gtrsim N^{-2s-1/2}(\log N)^{-1/2}\int_{N^2}^{N^2+N}\int_1^N \frac{d\xi_1}{\xi_1}d\xi \\ &= N^{-2s+1/2}(\log N)^{1/2}\end{aligned}$$

and

$$\begin{aligned}&\int_{N^2}^{N^2+N}\left|\int e^{-it\xi_1}\frac{\sin(t\xi_1)}{\xi_1}\widehat{a_{0,N}}(\xi_1)\widehat{u}(\xi-\xi_1)d\xi_1\right| d\xi \\ &\lesssim tN^{-2s-1/2}(\log N)^{-1/2}\int_{N^2}^{N^2+N}\int_1^N \frac{1}{\xi_1^2}\chi_{[N^2-N,N^2+N]}(\xi-\xi_1)d\xi_1 d\xi \\ &\le tN^{-2s+1/2}(\log N)^{-1/2},\end{aligned}$$

we get

$$\begin{aligned}
&\|\psi_N^{(2)}(t)\|_{H^s} \geq \|u_N^{(2)}\|_{H^s} \gtrsim N^{2s-1/2}\|\widehat{u_N^{(2)}}\|_{L^1_\xi(N^2<\xi<N^2+N)} \\
&\gtrsim N^{2s-1/2}\int_{N^2}^{N^2+N}\left(t|\widehat{a_{0,N}} * \widehat{u_N}(\xi)| - \left|\int e^{-it\xi}\frac{\sin(t\xi_1)}{\xi_1}\widehat{a_{0,N}}(\xi_1)\widehat{u_N}(\xi-\xi_1)d\xi_1\right|\right)d\xi \\
&\gtrsim t(\log N)^{1/2} - t(\log N)^{-1/2} \gtrsim t(\log N)^{1/2}.
\end{aligned}$$

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